DISCRETE U-DUALITY GROUPS

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Abstract Generators for the discrete U-duality groups of toroidally compactified M-theory in $d \geq 4$ are presented and used to determine the $d = 3$ U-duality group. This contribution summarizes the results of [1].

1. INTRODUCTION AND SUMMARY

The study of nonperturbative duality symmetries in the last years has dramatically changed our understanding of string theory. U-duality, introduced in [2], is one of these symmetries. It was conjectured to be generated by the target-space duality of T^d (see [4] for a review) and the modular group of the torus including the eleventh M-theory direction [5]. This definition was adopted in the algebraic approach to U-duality reviewed in [3].

Using instead the original conjecture of [2] that takes the hidden symmetries of low energy supergravity as a starting point, generators of discrete U-duality in four dimensions may be determined directly. Groups in higher dimensions can be found by embeddings, and a method to determine the $d = 3$ U-duality is presented. Applied to a toy model corresponding to a truncation of M-theory, the method is seen to give a significantly different result than for the full theory.

2. U-DUALITY IN FOUR DIMENSIONS

Compactifying eleven dimensional supergravity, the low energy limit of M-theory, on T^7 yields 70 scalars arising from the $d = 11$ 3-form potential and the moduli of the torus. They may be joint in a field $\mathcal{V}^{(4)} \in E_{7(+7)}/SU(8)$ [7], a representation matrix in the fundamental **56** representation. Furthermore, 28 $U(1)$ gauge fields arise from the

$d = 11$ potential and metric. These fields and their magnetic duals can be arranged to 28 dimensional vectors $\mathcal{G}_{\bar\mu\bar\nu}$ and $\mathcal{H}_{\bar\mu\bar\nu}$, $\bar\mu, \bar\nu = 0 \ldots 3$.

The equations of motion of the theory are invariant under classical $E_{7(+7)}$ duality and local $SU(8)$ transformations, acting as

$$\mathcal{F}_{\bar\mu\bar\nu} \to \Lambda^{-1}\mathcal{F}_{\bar\mu\bar\nu}, \quad \mathcal{V}^{(4)} \to h\,\mathcal{V}^{(4)}\Lambda, \quad \Lambda \in E_{7(+7)}, \quad h \in SU(8)$$

where $\mathcal{F}^t = (\mathcal{G}_{\bar\mu\bar\nu}, \mathcal{H}_{\bar\mu\bar\nu})^t$. Defining charges $\mathcal{Z} = \oint_\Sigma \mathcal{F}$, the DSZ quantization condition breaks $E_{7(+7)}$ to a discrete subgroup inducing integer shifts on the charge lattice. This group is the U-duality group and was proposed to extend to a nonperturbative quantum symmetry of M-theory.

To make U-duality transformations "manageable" in $d = 4$, the **56** representation needs to be addressed. This can be done by an embedding into $\mathfrak{e}_{8(+8)}$ using Freudenthal's realization of exceptional Lie algebras [8]. The $\mathfrak{e}_{8(+8)}$ generators are given by $E^i{}_j$, $\;i, j = 1 \ldots 9$, corresponding to $\mathfrak{sl}_9$, and E^{ijk}, E^*_{ijk}. Their commutators are given in [1]. Defining the basis

$$\mathcal{S}^t = \Big(-E^*_{\bar i\bar j 9}, +E^1{}_{\bar i} \mid -E^{1\bar i\bar j}, -E^{\bar i}{}_9 \Big), \quad \mathcal{X}^t = \Big(x^{\bar i\bar j}, x^{\bar i 9} \mid y_{\bar i\bar j}, y_{\bar i 9} \Big), \tag{1.1}$$

$\bar i, \bar j = 2 \ldots 8$, the adjoint action of the $\mathfrak{e}_{7(+7)}$ subalgebra on $\mathcal{X}\mathcal{S}$ exactly spans the **56** representation as defined in [7]. The basis (1.1) can therefore be used to study U-duality transformations on $\mathcal{F}$ and $\mathcal{V}^{(4)}$, as well as its subgroups T-duality and S-duality.

What are the generators of $E_{7(+7)}(\mathbb{Z})$? Using the fact that the **56** representation of $E_{7(+7)}$ is the unique minimal representation of $E_{7(+7)}$, it may be proven using the Birkhoff decomposition of Lie groups that the subgroup of $E_{7(+7)}$ inducing integer shifts on the lattice defined by $\mathcal{S}$ is generated by "fundamental unipotents"[1], that is, the action of the discrete subgroup is spanned by exponentiating the Chevalley generators for all positive and negative roots. From these, T and S generators may be built parallel to $SL(2, \mathbb{Z})$, the latter carrying a representation of the Weyl group modulo $\mathbb{Z}_2$. This together with the basis $\mathcal{S}$ yields contact to the algebraic approach to M-theory, and it may be shown that the two approaches are equivalent.

Since the notion of the above generators is representation independent, the discrete U-duality groups of higher dimensional theories follow directly from truncating the Dynkin diagram. Their representations are minimal and can be read off from $\mathcal{S}$.

[1] See [9], where this group is defined in a more general context as homeomorphism of the group ring over $\mathbb{Z}$ to $\mathbb{Z}$.

2.1 U-DUALITY IN THREE DIMENSIONS

The $d = 3$ theory is known to have classical $E_{8(+8)}$ symmetry. As only scalars remain in the theory, the notion of electric charge seems ill defined and the meaning of a duality symmetry seems unclear. Therefore, in order to define U-duality in $d = 3$, a method proposed in [2] parallel to [6] may be extended to M-theory. By compactifying M-theory on the torus, we can choose eight different ways how to compactify first to four dimensions. This results in eight $E_{7(+7)}(\mathbb{Z})$'s acting *differently* on M-theory fields. By reducing the theory further to three dimensions, these groups are merged together to form the three dimensional duality group.

The reduction to $d = 3$ yields a scalar coset matrix of the form

$$\mathcal{V}^{(3)} = \mathcal{V}'^{(4)} \exp\Big(\frac{1}{2}\phi \sum_{i=1}^{8} h_i\Big) \exp\Big(\mathcal{Y} \cdot \mathcal{S}\Big) \exp\Big(f \ E^{1}{}_{9}\Big). \tag{1.2}$$

$\mathcal{V}'^{(4)}$ is identical to $\mathcal{V}^{(4)}$, but now in the **248** adjoint representation of $E_{8(+8)}$. $\mathcal{Y}$ obeys $\partial_\mu \eta = \mathcal{G}_{\mu z}, \quad \partial_\mu \bar{\eta} = \mathcal{H}_{\mu z}, \quad \mathcal{Y} = (\eta, \bar{\eta})^t, \quad \mu, \nu = 0 \ldots 2$ and therefore carries the $d = 4$ charges, where z is the compact fourth direction. φ and f are the $d = 3$ dilaton and dualized KK field strength respectively, and h_i represents the $\mathfrak{sl}_9$ Cartan subalgebra.

A $\Lambda \in E_{7(+7)} \subset E_{8(+8)}$ transformation acts on $\mathcal{V}'^{(4)}$ and $\mathcal{Y}$ exactly as discussed in the last section. It therefore represents the $d = 4$ U-duality in $d = 3$. Completing a circle around vortex solutions in $d = 3$ may be seen to correspond to an $E_{7(+7)}(\mathbb{Z})$ action on the fields, parallel to [6].

The different compactifications yield 8 different coset matrices in $d = 3$. Using explicitly the connection to M-theory fields, it can be seen that they are related by

$$\mathcal{V}^{(3)}_{\#n} = (P_n S^{1}{}_{n})^{-1} \ h_n \ \ \mathcal{V}^{(3)}_{\#1} \ \ P_n S^{1}{}_{n}$$

where h_n is a local transformation restoring upper triangular parameterization, $S^{1}{}_{n}$ corresponds to a Weyl reflection and P_n corresponds to a charge conjugation in $d = 4$. Denoting the $d = 4$ U-duality generators of the nth compactification by Λ, the total $d = 3$ U-duality is given by joining all generators

$$P_n S^{1}{}_{n} \ \Lambda \ (P_n S^{1}{}_{n})^{-1}$$

for all n. This can be seen to yield the whole $E_{8(+8)}(\mathbb{Z})$ defined by exponentiating all Chevalley generators. The intersection of two different $d = 4$ U-dualities is seen to be $E_{6(+6)}(\mathbb{Z})$ as expected. This determines U-duality in $d = 3$.

2.2 $G_{2(+2)}$ AS TOY MODEL

The described method to generate $d = 3$ U-duality can also be applied to five dimensional simple supergravity as toy model, which upon reduction to three dimensions exhibits a $G_{2(+2)}$ global symmetry [10]. It is known that this theory closely resembles $d = 11$ supergravity in many respects and corresponds to a direct truncation. The analogue of U-duality in $d = 4$ is $SL(2, \mathbb{Z})$, acting in the spin 3/2 representation.

Here, the joint U-duality $U(\mathbb{Z})$ in $d = 3$ is strictly smaller than $G_{2(+2)}(\mathbb{Z})$ defined by exponentiating the Chevalley generators for all roots. All generators of $U(\mathbb{Z})$ correspond to short roots of $G_{2(+2)}(\mathbb{Z})$. That the groups do not agree is therefore connected to the fact that $G_{2(+2)}$ is not simply laced. Since no string compactification described by this no-moduli supergravity at low energies is known, one cannot determine which group is the "correct" U-duality group until such a description has been found.

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